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# Successive approximation of solutions to doubly perturbed stochastic differential equations with jumps

**Wei Mao**[1,2], **Liangjian Hu**[✉ 3], **Surong You**[3] and **Xuerong Mao**[4]

[1]College of Information Sciences and Technology, Donghua University, Shanghai, 201620, China
[2]School of mathematics and information technology, Jiangsu Second Normal University
Nanjing 210013, China
[3]Department of Applied Mathematics, Donghua Univerisity, Shanghai, 201620, China
[4]Department of Mathematics and Statistics, University of Strathclyde, Glasgow, G1 1XH, U.K.

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**Abstract.** In this paper, we study the existence and uniqueness of solutions to doubly perturbed stochastic differential equations with jumps under the local Lipschitz conditions, and give the $p$-th exponential estimates of solutions. Finally, we give an example to illustrate our results.

**Keywords:** doubly perturbed stochastic differential equations, Lévy jumps, local non-Lipschitz condition, $p$-th exponential estimates.

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## 1 Introduction

As the limit process from a weak polymers model, the following doubly perturbed Brownian motion

$$x_t = B_t + \alpha \max_{0\le s\le t} x_s + \beta \min_{0\le s\le t} x_s, \tag{1.1}$$

was studied by P. Carmona [4] and J. R. Norris [11]. Because of its important application, many people have devoted their investigation to this model and obtained a lot of results, for example, see [2,3,5–7,12,14]. Motivated by above mentioned works, R. A. Doney and T. Zhang [8] studied the singly perturbed Skorohod equations

$$x_t = x_0 + \int_0^t \sigma(x_s)dB_s + \int_0^t b(x_s)ds + \alpha \max_{0\le s\le t} x_s. \tag{1.2}$$

they proved the existence and uniqueness of the solution to equation(1.2) where the coefficients $b, \sigma$ satisfy the global Lipschitz conditions; Hu and Ren [9] and Luo [10] extended the global Lipschitz conditions of [8] to the case of non-Lipschitz conditions which are imposed by

✉Corresponding author. Email: ljhu@dhu.edu.cn

[13,17,18], they proved the existence and uniqueness of solutions to doubly perturbed neutral stochastic functional equations and doubly perturbed jump-diffusion processes, respectively.

However, for many practical situations, the nonlinear terms do not obey the global Lipschitz and linear growth condition, even the non-Lipschitz condition. For example, consider the singly perturbed semi-linear stochastic differential equations

$$dx(t) = ax(t)dt + \sigma(x(t))b(t,x(t))dB_t + \beta \max_{0\le s\le t} x(s), \qquad t \in [0,T]. \tag{1.3}$$

where $a \in R$, $\beta \in (0,1)$, and $\sigma(x)$ satisfies the local Lipschitz condition: For any integer $N > 0$, there exists a positive constant $k_N$ such that for all $x, y \in R$ with $|x|, |y| \le N$, it follows that

$$|\sigma(x) - \sigma(y)| \le k_N |x - y|. \tag{1.4}$$

Let us take

$$\rho(u) = \begin{cases} 0, & u = 0, \\ u[\log(u^{-1})]^r, & u \in (0,\delta], \\ \delta[\log(\delta^{-1})]^r, & u \in [\delta, +\infty], \end{cases}$$

where $r \in [0, \frac{1}{2})$ and $\delta \in (0,1)$ is sufficiently small, Assume $b(t,x)$ satisfies the non-Lipschitz condition

$$|b(t,x) - b(t,y)| \le \rho(|x-y|). \tag{1.5}$$

From the analysis of Section 5, the coefficients of equation (1.3) do not satisfy the global Lipschitz condition [8] or non-Lipschitz condition [9, 10]. In other words, the main results of [8–10] do not apply to equation (1.3). Therefore, it is very important to establish the existence and uniqueness theory of perturbed stochastic differential equations under some weaker conditions. The purpose of this paper is to study the existence and uniqueness of solutions to equation (2.1) with the local non-Lipschitz coefficients. Meantime, we will give the $p$th exponential estimates and the $p$th moment continuity of solutions.

This paper is organized as follows. In Section 2, we first give some preliminaries and assumptions on equation (2.1). In Section 3, we state and prove our main results. While in Section 4, we show that the $p$th moment of solution will grow at most exponentially. As an application of the $p$th exponential estimates, we give the continuity of the $p$th moment of solutions. Finally, we give an example to illustrate the theory in Section 5.

## 2 Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets). Let $\{w(t)\}_{t\ge0}$ be a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{\bar{p} = \bar{p}(t), t \ge 0\}$ be a stationary $\mathcal{F}_t$-adapted and $R$-valued Poisson point process. Then, for $A \in \mathcal{B}(R - \{0\})$, here $\mathcal{B}(R - \{0\})$ denotes the Borel $\sigma$-field on $R - \{0\}$ and $0 \notin$ the closure of $A$, we define the Poisson counting measure $N$ associated with $\bar{p}$ by

$$N((0,t] \times A) := \#\{0 < s \le t, \bar{p}(s) \in A\} = \sum_{t_0 < s \le t} I_A(\bar{p}(s)),$$

where # denotes the cardinality of set $\{\cdot\}$. It is known that there exists a $\sigma$-finite measure $\pi$ such that

$$E[N((0,t] \times A)] = \pi(A)t, \quad P(N((0,t] \times A) = n) = \frac{\exp(-t\pi(A))(\pi(A)t)^n}{n!}.$$

This measure $\pi$ is called the Lévy measure. Moreover, by Doob–Meyer's decomposition theorem, there exists a unique $\{\mathcal{F}_t\}$-adapted martingale $\tilde{N}((0,t] \times A)$ and a unique $\{\mathcal{F}_t\}$-adapted natural increasing process $\hat{N}((0,t] \times A)$ such that

$$N((0,t] \times A) = \tilde{N}((0,t] \times A) + \hat{N}((0,t] \times A), \qquad t > 0.$$

Here $\tilde{N}((0,t] \times A)$ is called the compensated Lévy jumps and $\hat{N}((0,t] \times A) = \pi(A)t$ is called the compensator.

Let $p \geq 2$, $\mathcal{L}^p([a,b];R)$ denote the family of $\mathcal{F}_t$-measurable, $R$-valued process $f(t) = \{f(t,\omega)\}$, $t \in [a,b]$ such that $\int_a^b |f(t)|^p dt < \infty$. For $Z \in \mathcal{B}(R - \{0\})$, consider the following doubly perturbed stochastic differential equations (SDEs) with Lévy jumps

$$\begin{aligned} x(t) = x(0) &+ \int_0^t f(s,x(s))ds + \int_0^t g(s,x(s))dw(s) \\ &+ \int_0^t \int_Z h(s,x(s-),v)N(ds,dv) + \alpha \max_{0\leq s\leq t} x(s) + \beta \min_{0\leq s\leq t} x(s), \end{aligned} \tag{2.1}$$

where $\alpha, \beta \in (0,1)$, the initial value $x(0) = x_0 \in R$ and $f : [0,T] \times R \to R$, $g : [0,T] \times R \to R$, $h : [0,T] \times R \times Z \to R$ are both Borel-measurable functions. In this paper, we assume that Lévy jumps $N$ is independent of Brownian motion $w$ and the random variable $x_0$ is independent of $w$, $N$ and satisfies $E|x_0|^p < \infty$.

To obtain the main results, we suppose $\int_Z \pi(dv) = \pi(Z) < \infty$ and give the following conditions.

**Assumption 2.1.** For any $x, y \in R$ and $t \in [0,T]$, there exist two functions $k(.)$, $\rho(.)$ such that

$$\begin{gathered} |f(t,x) - f(t,y)| \vee |g(t,x) - g(t,y)| \leq \lambda(t)k(|x-y|), \\ \int_Z |h(t,x,v) - h(t,y,v)|^2 \pi(dv) \leq \lambda^2(t)\rho^2(|x-y|), \end{gathered}$$

where $\lambda(t) \in \mathcal{L}^2([0,T],R)$, and $k(u), \rho(u)$ are two concave nondecreasing functions such that $k(0) = \rho(0) = 0$ and $\int_{0^+} \frac{u}{k^2(u)+\rho^2(u)} du = \infty$.

**Assumption 2.2.** There exist two positive constants $K_1$, $K_2$ such that

$$\sup_{0\leq t\leq T} \{|f(t,0)| \vee |g(t,0)|\} \leq K_1, \qquad \sup_{0\leq t\leq T} \int_Z |h(t,0,v)|^2 \pi(dv) \leq K_2.$$

**Assumption 2.3.** The coefficients satisfy $|\alpha| + |\beta| < 1$.

**Assumption 2.4.** For any integer $N > 0$, there exist two positive constants $k_N$ and $\rho_N$ such that

$$|f(t,x) - f(t,y)| \vee |g(t,x) - g(t,y)| \leq \lambda(t)k_N(|x-y|),$$

and

$$\int_Z |h(t,x,v) - h(t,y,v)|^2 \pi(dv) \leq \lambda^2(t)\rho_N^2(|x-y|),$$

for any $x, y \in R$ with $|x|, |y| \leq N$. Here $k_N(u), \rho_N(u)$ are two concave and nondecreasing functions such that $k_N(0) = \rho_N(0) = 0$ and $\int_{0^+} \frac{u}{k_N^2(u)+\rho_N^2(u)} du = \infty$.

**Remark 2.5.** Clearly, Assumptions 2.1 and 2.2 imply the linear growth condition. Since $k(\cdot)$ is concave and $k(0) = 0$, we can find a pair of positive constants $a$ and $b$ such that

$$k(u) \le a + bu, \quad \text{for} \;\; u \ge 0.$$

Therefore, for any $x \in R$ and $t \in [0, T]$,

$$\begin{aligned} |f(t,x)| \vee |g(t,x)| &\le \lambda(t)k(|x|) + K_1, \\ &\le a\lambda(t) + K_1 + b\lambda(t)|x|. \end{aligned}$$

Similarly, we can obtain

$$\int_Z |h(t,x,v)|^2 \pi(dv) \le 4\lambda^2(t)a^2 + 2K_2 + 4\lambda^2(t)b^2|x|^2.$$

In the sequel, to prove our main results we recall the following two lemmas.

**Lemma 2.6** ([1]). *Let $k : R_+ \to R_+$ be a continuous, non-decreasing function satisfying $k(0) = 0$ and $\int_{0^+} \frac{ds}{k(s)} = +\infty$. Let $u(\cdot)$ be a Borel measurable bounded non-negative function defined on $[0, T]$ satisfying*

$$u(t) \le u_0 + \int_0^t v(s)k(u(s))ds), \qquad t \in [0, T]$$

*where $u_0 > 0$ and $v(\cdot)$ is a non-negative integrable function on $[0, T]$. Then we have*

$$u(t) \le G^{-1}\left( G(u_0) + \int_0^t v(s)ds \right),$$

*where $G(t) = \int_{t_0}^t \frac{du}{k(u)}$ is well defined for some $t_0 > 0$, and $G^{-1}$ is the inverse function of $G$.*

*In particularly, if $u_0 = 0$, then $u(t) = 0$ for all $t \in [0, T]$.*

**Lemma 2.7** ([15]). *Let $\phi : R_+ \times Z \to R^n$ and assume that*

$$\int_0^t \int_Z |\phi(s,v)|^p \pi(dv)ds < \infty, \qquad p \ge 2.$$

*Then, there exists $D_p > 0$ such that*

$$\begin{aligned} &E\left( \sup_{0\le t\le u} \left| \int_0^t \int_Z \phi(s,v)\tilde{N}(ds,dv) \right|^p \right) \\ &\quad \le D_p \left\{ E\left( \int_0^u \int_Z |\phi(s,v)|^2 \pi(dv)ds \right)^{\frac{p}{2}} + E\int_0^u \int_Z |\phi(s,v)|^p \pi(dv)ds \right\}. \end{aligned}$$

## 3 Existence and uniqueness theorem

In this section, we study the existence and uniqueness of solutions to doubly perturbed SDEs with Lévy jumps and the local non-Lipschitz coefficients.

Let us consider the following equation

$$\begin{aligned} x(t) = x(0) &+ \int_0^t f(s)ds + \int_0^t g(s)dw(s) + \int_0^t \int_Z h(s-,v)N(ds,dv) \\ &+ \alpha \max_{0\le s\le t} x(s) + \beta \min_{0\le s\le t} x(s), \end{aligned} \tag{3.1}$$

with the initial data $x(0) = x_0$ and $f \in \mathcal{L}^2([0, T]; R)$, $g \in \mathcal{L}^2([0, T]; R)$ and $h \in \mathcal{L}^2([0, T] \times Z; R)$.

**Proposition 3.1.** *Under Assumptions 2.2–2.3, Equation (3.1) has a unique solution $x(t)$ on $[0,T]$.*

The proof of Proposition 3.1 is given in the Appendix.

Now, we construct a successive approximation sequence using a Picard type iteration. Let $x^0(t)=x_0$, $t\in[0,T]$, define the following Picard sequence:

$$\begin{aligned} x^n(t) &= x(0)+\int_0^t f(s,x^{n-1}(s))ds+\int_0^t g(s,x^{n-1}(s))dw(s)\\ &\quad +\int_0^t\int_Z h(s,x^{n-1}(s-),v)N(ds,dv)+\alpha\max_{0\le s\le t}x^n(s)+\beta\min_{0\le s\le t}x^n(s). \end{aligned} \tag{3.2}$$

Obviously, according to proposition 3.1, the solution $x^n(t)$ of equation (3.2) exists.

In what follows, $C>0$ is a constant which can change its value from line to line.

**Lemma 3.2.** *Under Assumptions 2.1–2.3, there exists a constant $C_1>0$ such that for any $t\in[0,T]$*

$$E\max_{0\le t\le T}|x^n(t)|^2\le C_1. \tag{3.3}$$

*Proof.* For any $s\ge 0$, it follows that from (3.2)

$$\begin{aligned} |x^n(s)| &\le |x(0)|+\left|\int_0^s f(\sigma,x^{n-1}(\sigma))d\sigma\right|+\left|\int_0^s g(\sigma,x^{n-1}(\sigma))dw(\sigma)\right|\\ &\quad+\left|\int_0^s\int_Z h(\sigma,x^{n-1}(\sigma-),v)N(d\sigma,dv)\right|+|\alpha|\left|\max_{0\le\sigma\le s}x^n(\sigma)\right|+|\beta|\left|\min_{0\le\sigma\le s}x^n(\sigma)\right|\\ &\le |x(0)|+\left|\int_0^s f(\sigma,x^{n-1}(\sigma))d\sigma\right|+\left|\int_0^s g(\sigma,x^{n-1}(\sigma))dw(\sigma)\right|\\ &\quad+\left|\int_0^s\int_Z h(\sigma,x^{n-1}(\sigma-),v)N(d\sigma,dv)\right|+(|\alpha|+|\beta|)\max_{0\le\sigma\le s}|x^n(\sigma)|. \end{aligned} \tag{3.4}$$

Taking the maximal value on both sides of (3.4), by the Hölder inequality, the Doob's martingale inequality and Assumption 2.3, we have

$$\begin{aligned} &(1-|\alpha|-|\beta|)^2E\max_{0\le s\le t}|x^n(s)|^2\\ &\le E|x(0)|^2+E\max_{0\le s\le t}\left|\int_0^s f(\sigma,x^{n-1}(\sigma))d\sigma\right|^2+E\max_{0\le s\le t}\left|\int_0^s g(\sigma,x^{n-1}(\sigma))dw(\sigma)\right|^2\\ &\quad+E\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x^{n-1}(\sigma-),v)N(d\sigma,dv)\right|^2\\ &\le C\Bigg[E|x(0)|^2+E\int_0^t|f(s,x^{n-1}(s))|^2ds+E\int_0^t|g(s,x^{n-1}(s))|^2ds\\ &\quad+[8+T\pi(Z)]E\int_0^t\int_Z|h(s,x^{n-1}(s-),v)|^2\pi(dv)ds\Bigg]. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &E\max_{0\le s\le t}|x^n(s)|^2\\ &\le\frac{C}{(1-|\alpha|-|\beta|)^2}[E|x(0)|^2+2E\int_0^t\left[|f(s,x^{n-1}(s))-f(s,0)|^2+|g(s,x^{n-1}(s))-g(s,0)|^2\right]ds\\ &\quad+2[8+T\pi(Z)]E\int_0^t\int_Z|h(s,x^{n-1}(s-),v)-h(s,0,v)|^2\pi(dv)ds\\ &\quad+2E\int_0^t\left[|f(s,0)|^2+|g(s,0)|^2\right]ds+2[8+T\pi(Z)]E\int_0^t\int_Z|h(s,0,v)|^2\pi(dv)ds]. \end{aligned}$$

By Assumptions 2.1 and 2.2, we have

$$E\max_{0\le s\le t}|x^n(s)|^2 \le \frac{C}{(1-|\alpha|-|\beta|)^2}\Big\{E|x(0)|^2+4K_1^2T+2[8+T\pi(Z)]K_2T \\ +4E\int_0^t|\lambda(s)|^2k^2(|x^{n-1}(s)|)ds \\ +2[8+T\pi(Z)]E\int_0^t|\lambda(s)|^2\rho^2(|x^{n-1}(s-)|)ds\Big\}.$$

Then the Jensen inequality implies that

$$E\max_{0\le s\le t}|x^n(s)|^2 \le \frac{C}{(1-|\alpha|-|\beta|)^2}\Big\{E|x(0)|^2+4K_1^2T+2[8+T\pi(Z)]K_2T \\ +4\int_0^t|\lambda(s)|^2k^2((E|x^{n-1}(s)|^2)^{\frac{1}{2}})ds \\ +2[8+T\pi(Z)]\int_0^t|\lambda(s)|^2\rho^2((E|x^{n-1}(s)|^2)^{\frac{1}{2}})ds\Big\}.$$

Letting $\gamma(x)=k^2(x^{\frac{1}{2}})+\rho^2(x^{\frac{1}{2}})$, it follows that

$$E\max_{0\le s\le t}|x^n(s)|^2 \le \frac{C}{(1-|\alpha|-|\beta|)^2}\Big\{E|x(0)|^2+4K_1^2T+2[8+T\pi(Z)]K_2T \\ +2[8+T\pi(Z)]\int_0^t|\lambda(s)|^2\gamma(E|x^{n-1}(s)|^2)ds\Big\}. \tag{3.5}$$

By Assumption 2.1, we have that $\gamma$ is a non-decreasing continuous function, $\gamma(0)=0$ and $\int_{0+}\frac{1}{\gamma(x)}dx=\infty$. Since $\frac{k(x)}{x}$, $\frac{\rho(x)}{x}$, $k'_+(x)$ and $\rho'_+(x)$ are non-negative, non-increasing functions, we have that

$$\gamma'_+(x)=x^{-\frac{1}{2}}\left[k(x^{\frac{1}{2}})k'_+(x)+\rho(x^{\frac{1}{2}})\rho'_+(x)\right]$$

is a non-negative, non-increasing function, thus $\gamma$ is a non-negative, non-decreasing concave function. Since $\gamma(\cdot)$ is concave and $\gamma(0)=0$, we can find a pair of positive constants $a$ and $b$ such that

$$\gamma(u)\le a+bu,\quad \text{for } u\ge 0,$$

we obtain

$$E\max_{0\le s\le t}|x^n(s)|^2 \le \frac{C}{(1-|\alpha|-|\beta|)^2}\Big\{1+E|x(0)|^2+a\int_0^t|\lambda(s)|^2ds \\ +b\int_0^t|\lambda(s)|^2E\max_{0\le\sigma\le s}|x^{n-1}(\sigma)|^2ds\Big\}. \tag{3.6}$$

Set

$$r(t)=\left[\frac{C}{(1-|\alpha|-|\beta|)^2}(1+E|x(0)|^2+a\int_0^t|\lambda(s)|^2ds\right]e^{b\int_0^t|\lambda(s)|^2ds},$$

then $r(\cdot)$ is the solution to the following ordinary differential equation:

$$r(t)=\frac{C}{(1-|\alpha|-|\beta|)^2}\left\{1+E|x(0)|^2+a\int_0^t|\lambda(s)|^2ds+b\int_0^t|\lambda(s)|^2r(s)ds\right\}.$$

By recurrence, it is easy to verify that for each $n \geq 0$,

$$E \max_{0\leq s\leq t} |x^n(s)|^2 \leq r(t).$$

Since $r(t)$ is continuous and bounded on $[0, T]$, we have

$$E \max_{0\leq s\leq t} |x^n(s)|^2 \leq C_1 < +\infty,$$

for any $n \geq 1$. The proof is complete. □

**Lemma 3.3.** *Let Assumptions 2.1–2.3 hold, then $\{x^n(t)\}_{n\geq 1}$ defined by (3.2) is a Cauchy sequence.*

*Proof.* For any $n, m \geq 1$, we have

$$\begin{aligned}
&|X^n(t) - X^m(t)| \\
&\leq \left|\int_0^t \left[f(s, x^{n-1}(s)) - f(s, x^{m-1}(s))\right] ds\right| + \left|\int_0^t \left[g(s, x^{n-1}(s)) - g(s, x^{m-1}(s))\right] dw(s)\right| \\
&\quad + \left|\int_0^t \int_Z \left[h(s, x^{n-1}(s-), v) - h(s, x^{n-1}(s-), v)\right] N(ds, dv)\right| \\
&\quad + |\alpha| \left|\max_{0\leq s\leq t} x^n(s) - \max_{0\leq s\leq t} x^m(s)\right| + |\beta| \left|\min_{0\leq s\leq t} x^n(s) - \min_{0\leq s\leq t} x^m(s)\right| \\
&\leq \left|\int_0^t \left[f(s, x^{n-1}(s)) - f(s, x^{m-1}(s))\right] ds\right| + \left|\int_0^t \left[g(s, x^{n-1}(s)) - g(s, x^{m-1}(s))\right] dw(s)\right| \\
&\quad + \left|\int_0^t \int_Z \left[h(s, x^{n-1}(s-), v) - h(s, x^{n-1}(s-), v)\right] N(ds, dv)\right| \\
&\quad + (|\alpha| + |\beta|) \max_{0\leq s\leq t} |x^n(s) - x^m(s)|. \qquad (3.7)
\end{aligned}$$

Taking the maximal value on both sides of (3.7), by the Hölder inequality, the Doob's martingale inequality and Assumption 2.3, we have

$$\begin{aligned}
&(1 - |\alpha| - |\beta|)^2 E \max_{0\leq s\leq t} |x^n(s) - x^m(s)|^2 \\
&\leq E \max_{0\leq s\leq t} \left|\int_0^s \left[f(\sigma, x^{n-1}(\sigma)) - f(\sigma, x^{m-1}(\sigma))\right] d\sigma\right|^2 \\
&\quad + E \max_{0\leq s\leq t} \left|\int_0^s \left[g(\sigma, x^{n-1}(\sigma)) - g(\sigma, x^{m-1}(\sigma))\right] dw(\sigma)\right|^2 \\
&\quad + E \max_{0\leq s\leq t} \left|\int_0^s \int_Z \left[h(\sigma, x^{n-1}(\sigma-), v) - h(\sigma, x^{m-1}(\sigma-), v)\right] N(d\sigma, dv)\right|^2 \\
&\leq TE \int_0^t \left|f(s, x^{n-1}(s)) - f(s, x^{m-1}(s))\right|^2 ds + 4E \int_0^t \left|g(s, x^{n-1}(s)) - g(s, x^{m-1}(s))\right|^2 ds \\
&\quad + [8 + 2T\pi(Z)]E \int_0^t \int_Z \left|h(s, x^{n-1}(s-), v) - h(s, x^{m-1}(s-), v)\right|^2 \pi(dv)ds.
\end{aligned}$$

By Assumption 2.1 and Jensen's inequality, we get

$$\begin{aligned}
&E \max_{0\leq s\leq t} |x^n(s) - x^m(s)|^2 \\
&\leq \frac{1}{(1 - |\alpha| - |\beta|)^2} \left[(T + 4) \int_0^t |\lambda(s)|^2 k^2((E|x^{n-1}(s) - x^{m-1}(s)|^2)^{\frac{1}{2}}) ds \right. \\
&\qquad \left. + (8 + 2T\pi(Z)) \int_0^t |\lambda(s)|^2 \rho^2((E|x^{n-1}(s) - x^{m-1}(s)|^2)^{\frac{1}{2}}) ds\right].
\end{aligned}$$

Similar to (3.5), we obtain

$$E\max_{0\le s\le t}|x^n(s)-x^m(s)|^2\le\frac{12+T+2T\pi(Z)}{(1-|\alpha|-|\beta|)^2}\int_0^t|\lambda(s)|^2\gamma(E|x^{n-1}(s)-x^{m-1}(s)|^2)ds. \tag{3.8}$$

By the inequality (3.3) and Fatou's lemma, it is easily seen that

$$\begin{aligned}&\limsup_{n,m\to\infty}E(\max_{0\le s\le t}|x^n(s)-x^m(s)|^2)\\&\quad\le\frac{12+T+2T\pi(Z)}{(1-|\alpha|-|\beta|)^2}\int_0^t|\lambda(s)|^2\gamma\left(\limsup_{n,m\to\infty}E\max_{0\le\sigma\le s}|x^n(\sigma)-x^m(\sigma)|^2\right)ds.\end{aligned} \tag{3.9}$$

Owing to Lemma 2.6, we immediately get that

$$\limsup_{n,m\to\infty}E(\max_{0\le s\le t}|x^n(s)-x^m(s)|^2)=0,\quad\text{for all } t\in[0,T], \tag{3.10}$$

Then $\{x^n(t)\}_{n\ge1}$ is a Cauchy sequence. The proof is complete. □

Now, we state and prove our main results.

**Theorem 3.4.** *Let Assumptions 2.1–2.3 hold, then equation* (2.1) *has a unique solution* $x(t)$ *on* $[0,T]$.

*Proof.* According to (3.10), it follows that there exists $x(t)\in\mathcal{L}^2([0,T];R)$ such that

$$\lim_{n\to\infty}E\sup_{0\le s\le t}|x^n(s)-x(s)|^2=0.$$

Then the Borel–Cantelli lemma can be used to show that $x^n(t)$ converges to $x(t)$ almost surely uniformly on $[0,T]$ as $n\to\infty$. Taking limits on both sides of (3.2) and letting $n\to\infty$, we obtain that $x(t)$ is a solution of equation (2.1).

Now we devote to proving the uniqueness of equation (2.1). Suppose $x(t)$ and $y(t)$ are two solutions of equation (2.1) with initial value $x_0$ and $y_0$, we have

$$\begin{aligned}&|x(t)-y(t)|\\&\quad\le|x_0-y_0|+\left|\int_0^t[f(s,x(s))-f(s,y(s))]\,ds\right|+\left|\int_0^t[g(s,x(s))-g(s,y(s))]\,dw(s)\right|\\&\qquad+\left|\int_0^t\int_Z[h(s,x(s-),v)-h(s,y(s-),v)]N(ds,dv)\right|+|\alpha|\left|\max_{0\le s\le t}x(s)-\max_{0\le s\le t}y(s)\right|\\&\qquad+|\beta|\left|\min_{0\le s\le t}x(s)-\min_{0\le s\le t}y(s)\right|.\end{aligned}$$

Then, in the same way as the proof of (3.8) one can show that

$$E\max_{0\le s\le t}|x(s)-y(s)|^2\le C|x_0-y_0|^2+C\int_0^t|\lambda(s)|^2\gamma\left(E\left(\max_{0\le\sigma\le s}|x(\sigma)-y(\sigma)|^2\right)\right)ds$$

for $t\in[0,T]$. By Lemma 2.6, we get

$$E\max_{0\le s\le t}|x(s)-y(s)|^2\le G^{-1}\left[G(C|x_0-y_0|^2)+C\int_0^t|\lambda(s)|^2ds\right],$$

where $G(t)=\int_1^t\frac{ds}{\gamma(s)}$. In particular, if $x_0=y_0$, then

$$G(C|x_0-y_0|^2)=-\infty,\qquad G(C|x_0-y_0|^2)+C\int_0^t|\lambda(s)|^2ds=-\infty.$$

Obviously, $G$ is a strictly increasing function, then $G$ has an inverse function which is strictly increasing, and $G^{-1}(-\infty) = 0$. Finally, we obtain

$$E \max_{0\le s\le t} |x(s) - y(s)|^2 = 0,$$

for any $t \in [0, T]$ which implies the uniqueness. This completes the proof. □

**Example 3.5.** We define the functions $k(\cdot)$, $\rho(\cdot)$ by $k(u) = \sqrt{u}$ and

$$\rho(u) = \begin{cases} 0, & u = 0, \\ u\sqrt{\log(u^{-1})}, & u \in (0, e^{-2}], \\ C\left(u + \dfrac{1}{3e^2}\right), & u \in (e^{-2}, \infty) \end{cases}$$

where $C > 0$. Then $k(\cdot)$ and $\rho(\cdot)$ satisfy Assumption 2.1 in Theorem 3.4.

**Theorem 3.6.** *Let Assumptions 2.2–2.4 hold. Then, there exists a unique solution $\{x(t)\}_{0\le t\le T}$ to equation (2.1).*

*Proof.* Let $T_0 \in (0, T)$, for each $N \ge 1$, we define the truncation function $f_N(t, x)$ as follows:

$$f_N(t,x) = \begin{cases} f(t,x), & |x| \le N, \\ f\left(t, N\frac{x}{|x|}\right), & |x| > N, \end{cases}$$

and $g_N(t, x)$, $h_N(t, x, v)$ similarly. Then $f_N$, $g_N$ and $h_N$ satisfy Assumption 2.1 due to that the following inequality about $f_N$, $g_N$ and $h_N$ hold:

$$\begin{aligned} |f_N(t,x) - f_N(t,y)| \vee |g_N(t,x) - g_N(t,y)| &\le 2\lambda(t)k_N(|x-y|), \\ \int_Z |h_N(t,x,v) - h_N(t,y,v)|^2\pi(dv) &\le 2\lambda^2(t)\rho_N^2(|x-y|), \end{aligned}$$

where $x, y \in R$ and $t \in [0, T_0]$. Therefore, by Theorem 3.4, there exists a unique solution $x_N(t)$ and $x_{N+1}(t)$, respectively, to the following equations

$$\begin{aligned} x_N(t) = x(0) &+ \int_0^t f_N(s, x_N(s))ds + \int_0^t g_N(s, x_N(s))dw(s) \\ &+ \int_0^t \int_Z h_N(s, x_N(s-), v)N(ds, dv) + \alpha \max_{0\le s\le t} x_N(s) + \beta \min_{0\le s\le t} x_N(s), \\ x_{N+1}(t) = x(0) &+ \int_0^t f_{N+1}(s, x_{N+1}(s))ds + \int_0^t g_{N+1}(s, x_{N+1}(s))dw(s) \\ &+ \int_0^t \int_Z h_{N+1}(s, x_{N+1}(s-), v)N(ds, dv) + \alpha \max_{0\le s\le t} x_{N+1}(s) + \beta \min_{0\le s\le t} x_{N+1}(s). \end{aligned}$$

Define the stopping times

$$\begin{aligned} \sigma_N &:= T_0 \wedge \inf\{t \in [0, T] : |x_N(t)| \ge N\}, \\ \sigma_{N+1} &:= T_0 \wedge \inf\{t \in [0, T] : |x_{N+1}(t)| \ge N+1\}, \\ \tau_N &:= \sigma_N \wedge \sigma_{N+1}, \end{aligned}$$

where we set $\inf\{\phi\} = \infty$ as usual. Similar to (3.7), we obtain

$$
\begin{aligned}
&|x_{N+1}(t) - x_N(t)| \\
&\quad = \left|\int_0^t [f_{N+1}(s, x_{N+1}(s)) - f_N(s, x_N(s))]ds\right| + \left|\int_0^t [g_{N+1}(s, x_{N+1}(s)) - g_N(s, x_N(s))]dw(s)\right| \\
&\qquad + \left|\int_0^t \int_Z [h_{N+1}(s, x_{N+1}(s-), v) - h_N(s, x_N(s-), v)]N(ds, dv)\right| \\
&\qquad + (|\alpha| + |\beta|) \max_{0\le s\le t} |x_{N+1}(s) - x_N(s)|.
\end{aligned}
$$

Again the Hölder inequality, the Doob's martingale inequality imply that

$$
\begin{aligned}
&E \max_{0\le s\le t\wedge\tau_N} |x_{N+1}(s) - x_N(s)|^2 \\
&\quad \le \frac{1}{(1-|\alpha|-|\beta|)^2}\Big[5TE\int_0^{t\wedge\tau_N} |f_{N+1}(s, x_{N+1}(s)) - f_N(s, x_N(s))|^2 ds \\
&\qquad + 20E\int_0^{t\wedge\tau_N} |g_{N+1}(s, x_{N+1}(s)) - g_N(s, x_N(s))|^2 ds \\
&\qquad + 20[2 + T\pi(Z)]E\int_0^{t\wedge\tau_N}\int_Z |h_{N+1}(s, x_{N+1}(s-), v) - h_N(s, x_N(s-), v)|^2 \pi(dv)ds\Big]. \quad (3.11)
\end{aligned}
$$

Noting that for any $0 \le s \le \tau_N$,

$$
\begin{aligned}
f_{N+1}(s, x_N(s)) &= f_N(s, x_N(s)), \; g_{N+1}(s, x_N(s)) = g_N(s, x_N(s)), \\
h_{N+1}(s, x_N(s-), v) &= h_N(s, x_N(s-), v),
\end{aligned}
$$

we derive that

$$
\begin{aligned}
&E \max_{0\le s\le t\wedge\tau_N} |x_{N+1}(s) - x_N(s)|^2 \\
&\quad \le \frac{1}{(1-|\alpha|-|\beta|)^2}\Big[5TE\int_0^{t\wedge\tau_N} |f_{N+1}(s, x_{N+1}(s)) - f_{N+1}(s, x_N(s))|^2 ds \\
&\qquad + 20E\int_0^{t\wedge\tau_N} |g_{N+1}(s, x_{N+1}(s)) - g_{N+1}(s, x_N(s))|^2 ds \\
&\qquad + 20[2 + T\pi(Z)]E\int_0^{t\wedge\tau_N}\int_Z |h_{N+1}(s, x_{N+1}(s-), v) - h_{N+1}(s, x_N(s-), v)|^2 \pi(dv)ds\Big].
\end{aligned}
$$

Then it follows from Assumption 2.4 that

$$
\begin{aligned}
&E \max_{0\le s\le t} |x_{N+1}(s\wedge\tau_N) - x_N(s\wedge\tau_N)|^2 \\
&\quad \le \frac{5T + 20 + 20[2 + T\pi(Z)]}{(1-|\alpha|-|\beta|)^2}\int_0^{t\wedge\tau_N} |\lambda(s)|^2 \gamma_{N+1}(E|x_{N+1}(s) - x_N(s)|^2)ds \\
&\quad \le \frac{5T + 20 + 20[2 + T\pi(Z)]}{(1-|\alpha|-|\beta|)^2}\int_0^{t} |\lambda(s)|^2 \gamma_{N+1}\left(E\max_{0\le\sigma\le s} |x_{N+1}(\sigma\wedge\tau_N) - x_N(\sigma\wedge\tau_N)|^2\right)ds,
\end{aligned}
$$

where $\gamma_N(\cdot) = k_N^2(\cdot^{\frac{1}{2}}) + \rho_N^2(\cdot^{\frac{1}{2}})$. Obviously, by Assumption 2.4, we have that $\gamma_N(\cdot)$ is a non-negative, non-decreasing concave function, $\gamma_N(0) = 0$ and $\int_{0^+}\frac{1}{\gamma_N(x)}dx = \infty$. By using Lemma 2.6 again, it follows that

$$
E \sup_{0\le s\le t} |x_{N+1}(s\wedge\tau_N) - x_N(s\wedge\tau_N)|^2 = 0.
$$

Therefore, we obtain that

$$x_{N+1}(t) = x_N(t), \quad \text{for } t \in [0, T_0 \wedge \tau_N].$$

For each $\omega \in \Omega$, there exists an $N_0(\omega) > 0$ such that $0 < T_0 \le \tau_{N_0}$. Now define $x(t)$ by $x(t) = x_{N_0}(t)$ for $t \in [0, T_0]$. Since $x(t \wedge \tau_N) = x_N(t \wedge \tau_N)$, it follows that

$$\begin{aligned} x(t \wedge \tau_N) &= x(0) + \int_0^{t\wedge\tau_N} f_N(s, x_N(s))ds + \int_0^{t\wedge\tau_N} g_N(s, x_N(s))dw(s) \\ &\quad + \int_0^{t\wedge\tau_N}\int_Z h_N(s, x_N(s-), v)N(ds, dv) + \alpha \max_{0\le s\le t\wedge\tau_N} x_N(s) + \beta \min_{0\le s\le t\wedge\tau_N} x_N(s) \\ &= x(0) + \int_0^{t\wedge\tau_N} f(s, x(s))ds + \int_0^{t\wedge\tau_N} g(s, x(s))dw(s) \\ &\quad + \int_0^{t\wedge\tau_N}\int_Z h(s, x(s-), v)N(ds, dv) + \alpha \max_{0\le s\le t\wedge\tau_N} x(s) + \beta \min_{0\le s\le t\wedge\tau_N} x(s). \end{aligned}$$

Letting $N \to \infty$, then yields

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dw(s) \\ &\quad + \int_0^t\int_Z h(s, x(s-), v)N(ds, dv) + \alpha \max_{0\le s\le t} x(s) + \beta \min_{0\le s\le t} x(s), \quad t \in [0, T_0]. \end{aligned}$$

Since $T_0$ is arbitrary, then we have $x(t)$ is the solution of equation (2.1) on $[0, T]$. The proof is complete. □

## 4 *p*-th moment exponential estimates

In this section, we will give the $p$th exponential estimates of solutions to equation (2.1).

**Assumption 4.1.** Assume $\lambda(t) \in \mathcal{L}^p([0, T], R)$, $p > 2$. For any $x, y \in R$ and $t \in [0, T]$, there exists a function $\rho(\cdot)$ and a constant $K_3$ such that

$$\int_Z |h(t, x, v) - h(t, y, v)|^p \pi(dv) \le \lambda^p(t)\rho^p(|x - y|),$$
$$\sup_{0\le t\le T}\int_Z |h(t, 0, v)|^p \pi(dv) \le K_3,$$

where $\rho(.)$ is defined as Assumption 2.1.

**Remark 4.2.** In particular, we see clearly that if let $\rho(u) = Ku$, $L > 0$, then Assumption 4.1 reduces to the linear growth condition. That is, for any $x \in R$ and $t \in [0, T]$, we have

$$\int_Z |h(t, x, v)|^p \pi(dv) \le 2^{p-1}\lambda^p(t)K^p|x|^p + 2^{p-1}K_3.$$

**Theorem 4.3.** *Let Assumptions 2.1–2.3 and 4.1 hold, for any $p \ge 2$*

$$E \max_{0\le t\le T} |x(t)|^p \le (1 + CE|x(0)|^p + C_4T)e^{C_5\int_0^T |\lambda(t)|^p dt}, \tag{4.1}$$

*where $C_4$ and $C_5$ are two positive constants of the inequality (4.11).*

*Proof.* For any $t \geq 0$, it follows from (2.1) that

$$
\begin{aligned}
|x(t)| \leq |x(0)| &+ \left|\int_0^t f(s,x(s))ds\right| + \left|\int_0^t g(s,x(s))dw(s)\right| \\
&+ \left|\int_0^t \int_Z h(s,x(s-),v)N(ds,dv)\right| + (|\alpha|+|\beta|)\max_{0\leq s\leq t}|x(s)|. \qquad (4.2)
\end{aligned}
$$

Taking the maximal value on both sides of (4.2), by Holder's inequality, the Burkholder inequality and Assumption 2.3, we have

$$
\begin{aligned}
&(1-|\alpha|-|\beta|)^p E\max_{0\leq s\leq t}|x(s)|^p \\
&\leq 4^{p-1}\left[E|x(0)|^p + E\max_{0\leq s\leq t}\left|\int_0^s f(\sigma,x(\sigma))d\sigma\right|^p + E\max_{0\leq s\leq t}\left|\int_0^s g(\sigma,x(\sigma))dw(\sigma)\right|^p\right. \\
&\qquad \left. + E\max_{0\leq s\leq t}\left|\int_0^s \int_Z h(\sigma,x(\sigma-),v)N(d\sigma,dv)\right|^p\right].
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
E\max_{0\leq s\leq t}|x(s)|^p \leq C&\left[E|x(0)|^p + E\max_{0\leq s\leq t}\left|\int_0^s f(\sigma,x(\sigma))d\sigma\right|^p + E\max_{0\leq s\leq t}\left|\int_0^s g(\sigma,x(\sigma))dw(\sigma)\right|^p\right. \\
&\left. + E\max_{0\leq s\leq t}\left|\int_0^s \int_Z h(\sigma,x(\sigma-),v)N(d\sigma,dv)\right|^p\right]. \qquad (4.3)
\end{aligned}
$$

where $C = \frac{4^{p-1}}{(1-|\alpha|-|\beta|)^p}$. Using Hölder's inequality, we get

$$
\begin{aligned}
E\max_{0\leq s\leq t}\left|\int_0^s f(\sigma,x(\sigma))d\sigma\right|^p &\leq T^{p-1}E\int_0^t |f(s,x(s))|^p ds \\
&\leq T^{p-1}E\int_0^t |f(s,x(s))-f(s,0)|^p ds
\end{aligned}
$$

By the basic inequality

$$
|a+b|^p \leq \left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}\left(|a|^p + \frac{|b|^p}{\epsilon}\right), \qquad p>1,\ a,b\in R^n,
$$

for any $\epsilon > 0$, it follows that

$$
E\max_{0\leq s\leq t}\left|\int_0^s f(\sigma,x(\sigma))d\sigma\right|^p \leq T^{p-1}\left[1+\epsilon^{\frac{1}{p-1}}\right]^{p-1}E\int_0^t\left(|f(s,x(s))-f(s,0)|^p + \frac{|f(s,0)|^p}{\epsilon}\right)ds.
$$

By Assumptions 2.1, 2.2 and letting $\epsilon = K_1^{p-1}$, we obtain

$$
E\max_{0\leq s\leq t}\left|\int_0^s f(\sigma,x(\sigma))d\sigma\right|^p \leq T^{p-1}(1+K_1)^{p-1}E\int_0^t[\lambda^p(s)k^p(|x(s)|)+K_1]ds.
$$

In fact, because the function $k(\cdot)$ is concave and increasing, there must exist a positive number $L$ such that

$$
k^p(|x|) \leq L(1+|x|^p), \quad \text{for all } p\geq 2. \qquad (4.4)
$$

Hence,

$$E\max_{0\le s\le t}\left|\int_0^s f(\sigma,x(\sigma))d\sigma\right|^p \le LT^{p-1}(1+K_1)^{p-1}\int_0^t|\lambda(s)|^p(1+E|x(s)|^p)ds + T^pK_1(1+K_1)^{p-1}. \tag{4.5}$$

By using the Burkholder–Davis–Gundy inequality and the Hölder inequality, we have a positive real number $C_p$ such that the following inequality holds:

$$\begin{aligned} E\max_{0\le s\le t}\left|\int_0^s g(\sigma,x(\sigma))dw(\sigma)\right|^p &\le C_pE\left(\int_0^t|g(s,x(s))|^2ds\right)^{\frac{p}{2}} \\ &\le C_pT^{\frac{p}{2}-1}E\int_0^t|g(s,x(s))|^pds. \end{aligned}$$

By the similar arguments, we have

$$E\max_{0\le s\le t}\left|\int_0^s g(\sigma,x(\sigma))dw(\sigma)\right|^p \le LC_pT^{\frac{p}{2}-1}(1+K_1)^{p-1}\int_0^t|\lambda(s)|^p(1+E|x(s)|^p)ds + C_pT^{\frac{p}{2}}K_1(1+K_1)^{p-1}. \tag{4.6}$$

Now, we will estimate the fourth term of (4.3). Using the basic inequality $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$, we have

$$\begin{aligned} &E\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)N(d\sigma,dv)\right|^p \\ &\quad\le 2^{p-1}E\left(\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)\tilde{N}(d\sigma,dv)\right|^p\right) \\ &\qquad+2^{p-1}E\left(\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)\pi(dv)d\sigma\right|^p\right). \end{aligned} \tag{4.7}$$

By Lemma 2.7 and Hölder's inequality, it follows that,

$$\begin{aligned} &E\left(\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)\tilde{N}(d\sigma,dv)\right|^p\right) \\ &\quad\le D_p\left\{T^{\frac{p}{2}-1}E\int_0^t\left[\int_Z|h(s,x(s-),v)|^2\pi(dv)\right]^{\frac{p}{2}}ds+E\int_0^t\int_Z|h(s,x(s-),v)|^p\pi(dv)ds\right\} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &E\left(\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)\pi(dv)d\sigma\right|^p\right) \\ &\quad\le T^{p-1}[\pi(Z)]^{\frac{p}{2}}E\int_0^t\left[\int_Z|h(s,x(s-),v)|^2\pi(du)\right]^{\frac{p}{2}}ds. \end{aligned} \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.7), and by Assumption 4.1, we have

$$\begin{aligned} &E\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)N(d\sigma,dv)\right|^p \\ &\quad\le 2^{p-1}(D_pT^{\frac{p}{2}-1}+T^{p-1}[\pi(Z)]^{\frac{p}{2}})E\int_0^t\left[2|\lambda(s)|^2\rho^2(|x(s-)|)+2K_2\right]^{\frac{p}{2}}ds \\ &\qquad+2^{2p-2}D_pE\int_0^t[|\lambda(s)|^p\rho^p(|x(s-)|)+K_3]ds. \end{aligned}$$

Similar to (4.4), we have

$$\rho^p(|x|) \le L(1+|x|^p), \quad \text{for all } p \ge 2.$$

Hence,

$$E\left(\max_{0\le s\le t}\left|\int_0^s\int_Z h(\sigma,x(\sigma-),v)\pi(dv)d\sigma\right|^p\right) \le (C_2+C_3)L\int_0^t|\lambda(s)|^p(1+E|x(s)|^p)ds + (C_2K_2^{\frac{p}{2}}+C_3K_3)T, \tag{4.10}$$

where $C_2 = 2^{2p-2}[D_pT^{\frac{p}{2}-1}+T^{p-1}[\pi(Z)]^{\frac{p}{2}}]$, $C_3 = 2^{2p-2}D_p$. Substituting (4.5), (4.6) and (4.10) into (4.3), we deduce that

$$1+E\max_{0\le s\le t}|x(s)|^p \le 1+CE|x(0)|^p+C_4T+C_5\int_0^t|\lambda(s)|^p\left(1+E\max_{0\le\sigma\le s}|x(\sigma)|^p\right)ds, \tag{4.11}$$

where

$$C_4 = K_1(1+K_1)^{p-1}(T^{p-1}+C_pT^{\frac{p}{2}-1})+(C_2K_2^{\frac{p}{2}}+C_3K_3),$$
$$C_5 = (1+K_1)^{p-1}L(T^{p-1}+C_pT^{\frac{p}{2}-1})+(C_2+C_3)L.$$

Let

$$u(t)=1+E\max_{0\le s\le t}|x(s)|^p, \qquad u_0 = 1+CE|x(0)|^p+C_4T, \qquad v(s)=|\lambda(s)|^p \quad \text{and} \quad k(s)=s,$$

then it follows from Lemma 2.6 that

$$1+E\max_{0\le s\le t}|x(s)|^p \le G^{-1}[G(1+CE|x(0)|^p+C_4T)+C_5\int_0^t|\lambda(s)|^pds],$$

where $G(t)=\int_{t_0}^t\frac{1}{s}ds=\ln(\frac{t}{t_0})$, $t_0>0$. Obviously, $G^{-1}(t)=t_0e^t, t>0$. Hence,

$$\begin{aligned}1+E\max_{0\le s\le t}|x(s)|^p &\le t_0e^{\ln(\frac{1+CE|x(0)|^p+C_4T}{t_0})+C_5\int_0^t|\lambda(s)|^pds}\\ &\le t_0e^{\ln(\frac{1+CE|x(0)|^p+C_4T}{t_0})}e^{C_5\int_0^t|\lambda(s)|^pds}\\ &\le (1+CE|x(0)|^p+C_4T)e^{C_5\int_0^t|\lambda(s)|^pds},\end{aligned}$$

and the proof is complete. □

**Remark 4.4.** Let $M=\sup_{0\le t\le\infty}|\lambda(t)|^p<\infty$. From Theorem 4.3, we know that the $p$th moment will grow at most exponentially with exponent $C_5M$. This can also be expressed as

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t)|^p)\le C_5M. \tag{4.12}$$

The inequality (4.12) shows that the $p$th moment Lyapunov exponent should not be greater than $C_5M$.

As an application of Theorem 4.3, we give the continuity of the $p$th moment of the solution.

**Theorem 4.5.** *Let the assumptions of Theorem 4.3 hold. For any $0\le s<t\le T$ and $p\ge2$, there exists a positive constant $C$ such that*

$$E|x(t)-x(s)|^p\le C(t-s)^{\frac{p}{2}}.$$

*Proof.* Similar to [16, Theorem 4.1] and by Theorem 4.3, we easily obtain the proof. □

## 5 An example

Let us return to equation (1.3),

$$dx(t) = ax(t)dt + \sigma(x(t))b(t,x(t))dB_t + \beta \max_{0\le s\le t} x(s), \qquad t \in [0,T]. \tag{5.1}$$

Assume that $\sigma(x)$ and $b(t,x)$ be continuous in $x$ for each $x \in R$. That is, there exists a positive constant $C$ such that

$$\sup_{x\in R} |\sigma(x)| \vee \sup_{t\in[0,T],\ x\in R} |b(t,x)| \le C.$$

Let

$$\gamma_N(u) = C(\rho(u) + k_N u),$$

then for any $x, y \in R$ and $|x| \vee |y| \le N$, we obtain that

$$\begin{aligned}|\sigma(x)b(t,x) - \sigma(y)b(t,y)| &\le \sup_{|x|\le N} |\sigma(x)||b(t,x) - b(t,y)| + \sup_{t\in[0,T],\ |x|\le N} |b(t,x)||\sigma(x) - \sigma(y)| \\ &\le C(\rho(|x-y|) + k_N|x-y|) \\ &\le \gamma_N(|x-y|).\end{aligned}$$

Since $\rho(u)$ is a concave function, it follows that $\gamma_N(u) = C(\rho(u) + k_N u)$ is a concave non-decreasing function with $\gamma_N(0) = 0$ and $\rho(u) \ge \rho(1)u$ for $0 \le u < 1$. Hence, we have

$$\begin{aligned}\int_{0^+} \frac{u}{\gamma_N^2(u)} du &= \int_{0^+} \frac{u}{(k_N u + \rho(u))^2} du = \int_{0^+} \left(\frac{\rho(u)}{k_N u + \rho(u)}\right)^2 \frac{u}{\rho^2(u)} du \\ &\ge \left(\frac{\rho(1)}{k_N + \rho(1)}\right)^2 \int_{0^+} \frac{u}{\rho^2(u)} du \\ &= \left(\frac{\rho(1)}{k_N + \rho(1)}\right)^2 \int_{0^+} \frac{1}{u(\log u^{-1})^{2r}} du \\ &= \left(\frac{\rho(1)}{k_N + \rho(1)}\right)^2 \frac{1}{2r-1} (\log u^{-1})^{1-2r}|_{0^+} = \infty.\end{aligned}$$

Clearly, the coefficients $ax, \sigma(x)b(t,x), \beta$ satisfy Assumptions 2.2–2.4, then by Theorem 3.6, we have that equation (5.1) has a unique solution $x(t)$ on [0,T].

## Appendix

*Proof of Proposition 3.1.* Let $\Phi$ be the function defined on $R$ by

$$\begin{aligned}\Phi(x)(t) := x(0) &+ \int_0^t f(s)ds + \int_0^t g(s)dw(s) \\ &+ \int_0^t \int_Z h(s-,v)N(ds,dv) + \alpha \max_{0\le s\le t} x(s) + \beta \min_{0\le s\le t} x(s).\end{aligned}$$

Clearly, $\Phi(x)$ is $R$-valued measurable $\{\mathcal{F}_t\}$-adapted process. we will first prove the mean square continuity of $\Phi$ on $[0,T]$. Let $r$ be sufficiently small, we have

$$\begin{aligned}\Phi(x)(t+r) - \Phi(x)(t) = &\int_t^{t+r} f(s)ds + \int_t^{t+r} g(s)dw(s) + \int_t^{t+r} \int_Z h(s-,v)N(ds,dv) \\ &+ \alpha \left[\max_{0\le s\le t+r} x(s) - \max_{0\le s\le t} x(s)\right] + \beta \left[\min_{0\le s\le t+r} x(s) - \min_{0\le s\le t} x(s)\right].\end{aligned}$$

By the basic inequality $|a+b+c+d+e|^2 \le 5|a|^2 + 5|b|^2 + 5|c|^2 + 5|d|^2 + 5|e|^2$, it follows that

$$
\begin{aligned}
& E|\Phi(x)(t+r) - \Phi(x)(t)|^2 \\
& \le 5E\left|\int_t^{t+r} f(s)ds\right|^2 + 5E\left|\int_t^{t+r} g(s)dw(s)\right|^2 + 5E\left|\int_t^{t+r}\int_Z h(s-,v)N(ds,dv)\right|^2 \\
& \quad + 5E\left|\alpha\left[\max_{0\le s\le t+r} x(s) - \max_{0\le s\le t} x(s)\right]\right|^2 + 5E\left|\beta\left[\min_{0\le s\le t+r} x(s) - \min_{0\le s\le t} x(s)\right]\right|^2. \qquad (5.2)
\end{aligned}
$$

Using the Hölder inequality, we obtain

$$E\left|\int_t^{t+r} f(s)ds\right|^2 \le rE\int_t^{t+r} |f(s)|^2 ds.$$

It is easy to obtain that $5E\left|\int_t^{t+r} f(s)ds\right|^2 \to 0$, as $r \to 0$. Furthermore, the martingale isometry implies

$$5E\left|\int_t^{t+r} g(s)dw(s)\right|^2 = 5E\int_t^{t+r} |g(s)|^2 ds \to 0$$

as $r \to 0$. For the third term of (5.2), by using the basic inequality $|a+b|^2 \le 2(|a|^2 + |b|^2)$, we have

$$
\begin{aligned}
5E\left|\int_t^{t+r}\int_Z h(s-,v)N(ds,dv)\right|^2 & \le 10E\left|\int_t^{t+r}\int_Z h(s-,v)\tilde{N}(ds,dv)\right|^2 \\
& \quad + 10E\left|\int_t^{t+r}\int_Z h(s-,v)\pi(dv)ds\right|^2,
\end{aligned}
$$

where $N(dt,dv) = \tilde{N}(dt,dv) + \pi(dv)dt$. By Hölder's inequality and the martingale isometry, we derive that

$$
\begin{aligned}
E\left|\int_t^{t+r}\int_Z h(s-,v)\pi(dv)ds\right|^2 & \le E\int_t^{t+r} ds \int_t^{t+r}\left|\int_Z h(s-,v)\pi(dv)\right|^2 ds, \\
& \le \int_t^{t+r} ds \int_Z \pi(dv) \int_t^{t+r}\int_Z E|h(s-,v)|^2\pi(dv)ds,
\end{aligned}
$$

and

$$E\left|\int_t^{t+r}\int_Z h(s-,v)\tilde{N}(ds,dv)\right|^2 = E\int_t^{t+r}\int_Z |h(s-,v)|^2\pi(dv)ds.$$

Therefore,

$$5E\left|\int_t^{t+r}\int_Z h(s-,v)N(ds,dv)\right|^2 \le 10[1+r\pi(Z)]E\int_t^{t+r}\int_Z |h(s-,v)|^2\pi(dv)ds \to 0,$$

as $r \to 0$. Obviously, we have

$$5E\left|\alpha\left[\max_{0\le s\le t+r} x(s) - \max_{0\le s\le t} x(s)\right]\right|^2 + 5E\left|\beta\left[\min_{0\le s\le t+r} x(s) - \min_{0\le s\le t} x(s)\right]\right|^2 \to 0$$

as $r \to 0$. Therefore, $\Phi$ is mean square continuous on $[0,T]$.

Next, we will claim $\Phi(R) \subset R$. That is to say, if $E(\sup_{0\le t\le T} |x(t)|^2) < \infty$, then

$$E\left(\sup_{0\le t\le T} |\Phi(x)(t)|^2\right) < \infty.$$

Using the basic inequality, it follows that

$$E\left(\sup_{0\le t\le T}|\Phi(x)(t)|^2\right)\le 6E|x_0|^2+6E\sup_{0\le t\le T}\left|\int_0^t f(s)ds\right|^2+6E\sup_{0\le t\le T}\left|\int_0^t g(s)dw(s)\right|^2$$
$$+6E\sup_{0\le t\le T}\left|\int_0^t\int_Z h(s-,v)N(ds,dv)\right|^2+6E\sup_{0\le t\le T}\left|\alpha\max_{0\le s\le t}x(s)\right|^2$$
$$+6E\sup_{0\le t\le T}\left|\beta\min_{0\le s\le t}x(s)\right|^2.$$

By the Hölder inequality and Doob's martingale inequality, we obtain

$$E\left(\sup_{0\le t\le T}|\Phi(x)(t)|^2\right)\le 6E|x_0|^2+6TE\int_0^T|f(s)|^2ds+24E\int_0^T|g(s)|^2ds$$
$$+24E\int_0^T\int_Z|h(s-,v)|^2\pi(dv)ds+6(\alpha^2+\beta^2)E\sup_{0\le t\le\tau_1\wedge T}|x(t)|^2.$$

Since $f\in\mathcal{L}^2([0,T];R)$, $g\in\mathcal{L}^2([0,T];R)$ and $h\in\mathcal{L}^2([0,T]\times Z;R)$, it follows that

$$E(\sup_{0\le t\le T}|\Phi(x)(t)|^2)<\infty. \tag{5.3}$$

Hence, (5.3) implies $\Phi$ is a operator from $\mathcal{L}^2([0,T];R)$ to itself and we conclude that $\Phi$ is well defined.

Now, we prove that $\Phi$ has a unique fixed point. For any $x,y\in\mathcal{L}^2([0,T];R)$, we have

$$E\sup_{0\le t\le T}|\Phi(x)(t)-\Phi(y)(t)|^2\le E\sup_{0\le t\le T}|\alpha\max_{0\le s\le t}[x(s)-y(s)]+\beta\min_{0\le s\le t}[x(s)-y(s)]|^2$$
$$\le(|\alpha|+|\beta|)^2E\sup_{0\le t\le T}|x(t)-y(t)|^2.$$

By Assumption 2.3, we have $(|\alpha|+|\beta|)^2<1$, then the operator $\Phi$ is a contraction mapping on $R$ and therefore has a unique fixed point in $\mathcal{L}^2([0,T];R)$ which is a solution of equation (3.1) on $[0,T]$, i.e., there exists a unique stochastic process $x=x(t)$ satisfying

$$E\sup_{0\le t\le T}|\Phi(x)(t)-x(t)|^2=0.$$

So $x(t)$ is a unique solution of equation (3.1) in $[0,T]$. The proof is complete. □

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